Indian Statistical Institute, Bangalore

B. Math.

First Year, First Semester

Analysis I

Mid-term examination — Date : Sept. 9, 2019

Total Marks: 105 Maximum marks: 100 — Time: 3 hours

Notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$-the set of natural numbers. $\mathbb{R}$-the set of real numbers.

(1) Show that the set of functions $f : \mathbb{N} \to \{1, 2\}$ satisfying $f(i) \leq f(j)$ for all $1 \leq i \leq j$, is countable. [15]

(2) Show that the set of natural numbers is unbounded. Use this and prove the Archimedean property for real numbers. [15]

(3) Let $\{h_n\}_{n\geq 1}$ be the sequence of real numbers defined recursively by: $h_1 = 1$ and $h_n = h_{n-1} + \frac{1}{n}$ for $n \geq 2$. Show that the sequence $\{h_n\}_{n\geq 1}$ is not bounded. (Hint: $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$; $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ etc.) [15]

(4) Let $\{c_n\}_{n\geq 1}$ be a sequence of non-zero real numbers converging to a non-zero real number a. Show that $\frac{1}{c_n}$ converges to $\frac{1}{a}$. [15]

(5) Let $\{x_n\}_{n\geq 1}$ be a bounded sequence and L be its set of limit points. Let $\{a_n\}_{n\geq 1}$ be a sequence with $a_n \in L$ for every n. Suppose $\{a_n\}_{n\geq 1}$ converges to b. Show that $b \in L$. [15]

(6) Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of real numbers satisfying $a_n \geq \frac{1}{4}$ for all $n \geq 1$. Prove or disprove the following statements:
   (i) $\limsup_{n\to\infty}(a_{n+1} - a_n) = 0$.
   (ii) $\limsup_{n\to\infty}(a_n^2) = (\limsup_{n\to\infty} a_n)^2$.
   (iii) $\limsup_{n\to\infty}(\frac{1}{a_n}) = \frac{1}{\liminf_{n\to\infty} a_n}$.

[15]

(7) Obtain a bounded sequence whose set of limit points is given by

$$\{0\} \bigcup \{\frac{1}{n} : n \geq 1\}.$$

Prove your claim. [15]